UNDERSTANDING PROOF AND TRANSFORMING TEACHING

David A Reid
Acadia University
david.reid@acadiau.ca

This article reviews some key ideas about the nature of proof, kinds of proofs, and the way of reasoning called proving. This provides background for a discussion of the teaching of proof, in which some important aspects of teaching that support students' proving are described, and the ideas of a tool-box of accepted premises and local organization of knowledge are explored as ways to structure teaching. Finally, lessons learned from approaches to teaching proof and problem solving in the past are connected with the earlier discussions of understanding and teaching proof to suggest a transformation of proof teaching into proof based teaching, in which proof is a process of coming to understanding rather than a topic to be taught and learned.

The Reasoning and Proof Standard (National Council of Teachers of Mathematics [NCTM], 2000) advocates that all students should recognize proof as fundamental to mathematics, should read and write mathematical proofs, and should reason in various ways, including the way called *proving*. In this article I will consider what these goals might mean, and how they might be achieved. I will begin by considering the nature of proof and four ways in which it is fundamental to mathematics: as a means to verify, explain, discover and belong. I will give some examples of proofs and compare their formats and functions. And I will examine the types of reasoning involved in conjecturing and proving. This will provide the background for a consideration of the teaching of proof. I will describe aspects of teaching that seem to create a good context for proving in school, and two ideas, a tool-box of accepted premises and local organization of knowledge, that have potential as ways to structure teaching. Finally, I will bring together my discussions of the nature of proof, proofs and proving and the teaching of proof to suggest a possible role for proof in mathematics teaching, as a way to teach rather than a topic to be taught.

Proof

What does it mean to recognize proof as a fundamental aspect of mathematics? How is proof fundamental? It is clear that proof is essential to the work of professional mathematicians. Papers without proofs do not get published. But why are proofs needed in mathematics?

Verification

Fischbein and Kedem (1982) say that proofs are needed because "a formal proof of a mathematical statement confers on it the attribute of a priori universal validity" (p. 128). In other words, proof tells us what is true and what is not. This idea, that mathematical proofs verify mathematical statements, has been recognized and admired for a long time. For example, Descartes commented:

> Of all those who have already searched for truth in the sciences, only the mathematicians were able to find demonstrations, that is, *certain* and evident reasons. (Descartes, 1637/1993, p. 11, italics added)

Mathematical proofs provide certainty. That is one answer why they are needed and why proof is a fundamental aspect of mathematics.

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

While many people believe that the essential role of proof in mathematics is verification of theorems, there are objections to this position. Some are fairly abstract, like the observation that Gödel's Incompleteness Theorem tells us that there are true statements in mathematics that have no proofs. Others are completely pragmatic, like Davis's (1972) comment that "most proofs in research papers are unchecked other than by the author ... They are loaded with errors." (p. 259). Obviously proofs with mistakes in them cannot tell us what is true and what is not in any absolute way. Davis concludes that "A derivation of a theorem or a verification of a proof has only probabilistic validity." And there are objections based on looking at how mathematicians actually behave, as opposed to what they say they do. Gila Hanna (1983, p. 70), for example, lists five factors other than a mathematical proof that contribute to the acceptance of theorems, including the reputation of the author and the plausibility of the result.

Explanation

There are other roles proof plays that make proof fundamental to mathematics. "Proof, in its best instances, increases understanding by revealing the heart of the matter." (Davis & Hersh, 1981, p. 151). Proof helps us understand and explain mathematics. Especially in the context of education, this explanatory role has increasingly come to be seen as vital. Gila Hanna was perhaps the first to point out to mathematics educators, at PME 13 in Paris (1989, p. 2-45), that proofs that explain, that show not only that the statement is true but also why it is true, should be favored in mathematics education.

To understand this distinction, consider these two proofs, both of which show that the sum of the first n integers $S(n)$ is $n(n+1)\div 2$:

Initial step
For n = 1 it is true since $1 = 1(1 + 1) \div 2$.

Induction step
Assume it is true for some arbitrary k, that is,
$S(k) = k(k + 1) \div 2$.

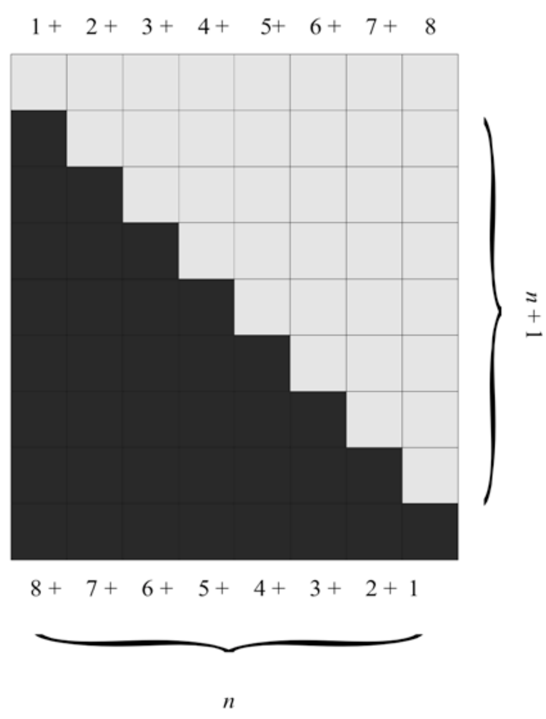
Then consider

$$\begin{aligned} S(k + 1) &= S(k) + (k + 1) \\ &= k(k + 1) \div 2 + (k + 1) \\ &= (k + 1)(k + 2) \div 2 \end{aligned}$$

Therefore, if the statement is true for k it is true for $k + 1$.

Hence, by mathematical induction, the statement is true for all n.
(adapted from Reid & Knipping, 2010, p. 99)

Proof 1



Proof 2

Both proofs require some unpacking, but of a very different kind. To understand Proof 1 the reader must be familiar with mathematical induction as a proof technique, and even then if the technique has been taught formally, without first establishing a link to informal ways of

reasoning recursively, the whole process can seem entirely arbitrary. In that case a proof like Proof 1 can be not only non-explanatory, but also unconvincing. Proof 2 offers an example. It does not show that $S(n)$ is $n(n+1)\div 2$. All it shows is that $S(8)$ is $8(8+1)\div 2$. The reader must see that this specific example is in fact a generic example; that the same process would work for any n. But once this is grasped, the proof is explanatory. It gives answers to questions like "Why do you divide by 2?" and "Why do you multiply by $n+1$?"

Discovery

Proof is also fundamental to mathematics as a way of discovering new knowledge. In mathematics education de Villiers seems to have been the first to point this out. In 1990 he noted:

> Proof can frequently lead to new results. To the working mathematician proof is therefore not merely a means of a posteriori verification, but often also a means of exploration, analysis, discovery and invention. (p. 21)

The role of proof in the discovery of non-Euclidean geometries is an historically significant example of proof as a means of discovery. Here is a more simple example:

There are many ways to prove that the sum of any two consecutive odd numbers is even. The simplest is to note that the sum of any two odd numbers (consecutive or not) is even. This verifies the statement and explains it. It is also possible to verify the statement with a proof using mathematical induction, if you believe in it, producing a proof that verifies but does not explain. You could use a generic example: 7+9 = 7+7+2, which must be even because 7+7 must be even and 2 is even. This proof both verifies and explains. Or you could do some algebra: $2n - 1 + 2n + 1 = 2(2n)$ which is even. The final proof lets you discover something more that you were trying to prove. $2(2n)$ is $4n$, so the sum of any two consecutive odd numbers is not only even, it is a multiple of four.

Being a Mathematician

Papers without proofs do not get published. That signals another role of proof in mathematics. Publishing proofs is part of being a mathematician. As Thurston (1995) points out:

> We are driven by considerations of economics and status. Mathematicians, like other academics, do a lot of judging and being judged. Starting with grades, and continuing through letters of recommendation, hiring decisions, promotion decisions, referees reports, invitations to speak, prizes...we are involved in a fiercely competitive system. (p. 34, ellipses in original) In our credit driven system, [mathematicians] also want and need *theorem credits* (p. 36, emphasis in original).

By "theorem credits" Thurston means the social acknowledgement that comes from publishing theorems, and in mathematics results must be published with proofs to count.

This social role of proof is usually implicit. As long as everyone conforms to the norm, it does not become evident. However, when a mathematician publishes without proving theorems, the reaction can be very strong. For example, when Benoit Mandelbrot named and began publishing images of fractals Steven Krantz (1989) published a critique in the *Mathematical Intelligencer* that focused on the lack of definitions, proofs and theorems in Mandelbrot's work. Without these, could fractal geometry be part of mathematics? The exclusion of non-Europeans from the history of mathematics, on the same basis that they did not prove their work (Joseph, 1991), provides another example.

Proofs

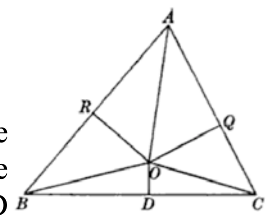
Let us turn now to proofs, as opposed to the concept of proof. What is a proof? The NCTM has a simple definition:

> By the end of secondary school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments. (2000, p. 56)

This definition is not far from the first ones published in the North American mathematics textbooks:

> Every statement in a proof must be based upon a postulate, an axiom, a definition, or some proposition previously considered of which the student is prepared to give the proof again when he refers to it. ... No statement is true simply because it appears to be true from a figure. ... [In a proof] are set forth, in concise steps, the statements to prove the conclusion ... asserted. (Beman and Smith, 1899, pp. 19–20, cited in Herbst, 2002)

A proof is something like this:



Given an arbitrary triangle ABC we will show that AB = AC.

Construction

Let O be the intersection of the angle bisector of ∠BAC and the perpendicular bisector of segment BC. (In the case where these lines are parallel it is easy to show AB = AC. This is left as an exercise.) Draw OD perpendicular to BC, OR perpendicular to AB, and OQ perpendicular to AC. Draw OB and OC.

Proof

1	OD = OD	reflexive property of equality
2	BD = CD	OD bisects BC, definition of perpendicular bisector
3	∠ODB ≅ ∠ODC	both are right angles, definition of perpendicular bisector
4	△ODB ≅ △ODC	Side Angle Side congruence property
5	OB = OC	Corresponding Parts of Congruent Triangles are Congruent
6	AO = AO	reflexive property of equality
7	∠OAQ ≅ ∠OAR	AO bisects ∠A, definition of angle bisector
8	∠ARO ≅ ∠AQO	both are right angles, definition of perpendicular bisector
9	△RAO ≅ △QAO	Angle Angle Side congruence property
10	AR = AQ & OR = OQ	Corresponding Parts of Congruent Triangles are Congruent
11	∠ORB ≅ ∠OQC	both are right angles, definition of perpendicular bisector
12	OB = OC	from 5
13	OR = OQ	from 10
14	△ROB ≅ △QOC	Hypotenuse Leg congruence property
15	RB = QC	Corresponding Parts of Congruent Triangles are Congruent
16	AB = AR + RB	segment addition postulate
17	AR + RB = AQ + QC	from 10 and 15, addition property of equality
18	AQ + QC = AC	segment addition postulate
19	AB = AC [Q.E.D.]	transitivity of equality

Proof 3 (adapted from Maxwell, 1959, Chapter II, § 1)

Every statement in the proof is based upon a postulate, an axiom, a definition, or some proposition previously proven. The conclusions follow from the hypotheses by logically rigorous deductions. To make it clear that this is happening, the proof is arranged in two columns, one

giving the concise steps of the proof, and the other providing the supporting postulates and so on. This is the famous two column proof format that was invented by two American textbook authors, Arthur Schultze and Frank Sevenoak in the early years of the twentieth century (Herbst, 2002, p. 297). According to Herbst, the two column proof format was an innovation that allowed teachers to better handle some of the challenges they faced. At that time a greater proportion of the population was attending secondary school and taking high school geometry, and at the same time the focus of high school geometry courses had shifted from the content of geometry to the process of proving. Explicit descriptions of what a proof should be (like that of Beman and Smith quoted above) and the general use of the two column format to present all proofs seen in schools, meant that the object of learning, proofs, was explicit. This meant that the teacher and students were clear about the object of the class.

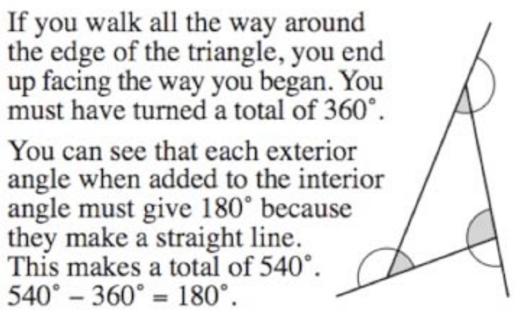
However, writing school mathematics proofs in the two column format means that they no longer resemble mathematicians' proofs. In mathematicians' proofs any step that the reader can be expected to supply is omitted, and explicit reference is made to supporting postulates, definitions and theorems only when they are unusual. If our goal is that "High school students should be able to present mathematical arguments in written forms that would be acceptable to professional mathematicians" (NCTM, 2000, p. 58) then we will have to start teaching written forms that are more like those of professional mathematicians.

Another disadvantage of the two column format is that it shifts the focus away from the statement being proven and towards the form and details of the proof. As a result, students are likely to believe that a proof like Proof 3 would be accepted, and even praised, by their teachers, in spite of the fact that its conclusion is false.

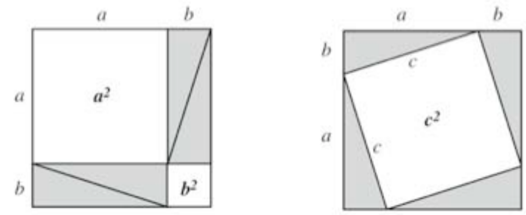
As well as illustrating the two column format, Proof 3 also allows us to revisit the question of the role of proof with a concrete example. Assuming you know that some triangles are not isosceles, you are unlikely to have been convinced by Proof 3. It does not verify anything. Nor is it particularly explanatory or useful for discovering new knowledge (unless you do find it convincing, and then it would lead you to the new knowledge that not only are all triangles isosceles, they are all equilateral, as the same reasoning can be applied to any two sides of the triangle.) What it might be useful for, from a student's point of view, is satisfying school norms for what a proof should look like. A student might expect, in the context of a tradition in which the two column format is highly valued, to be rewarded for producing such a proof. Proof 3 also illustrates a pitfall in Beman and Smith's description of a proof. They require that "No statement is true simply because it appears to be true from a figure." This reflects a general distrust of reasoning using pictures that was dominant in the twentieth century. But pictures can be extremely useful. The simplest way to see what is wrong with Proof 3 is not to go through it line by line looking for a missing or unjustified step. The simplest way to is draw a picture, as accurately as you can, following the steps of the initial construction.

As we have seen above, visual proofs can be both convincing and explanatory, especially in school contexts. Proof 4 (from Hoyles, 1997, p. 12) is another example, and one that works even better as an action proof if you trace a triangle on the floor and walk around it, paying attention to the way that your body turns.

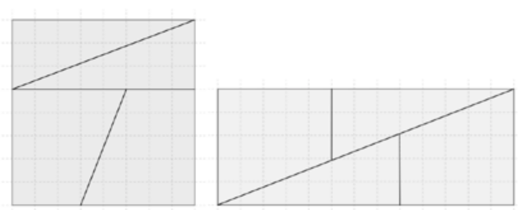
Another well known visual proof is Proof 5, of the Pythagorean Theorem. This kind of proof reflects a very old Chinese tradition of proving using dissections of shapes. Occasionally, such proofs are rejected because of the existence of false proofs like Proof 6, which shows that 64=65. But this makes no more sense than rejecting all two column proofs because Proof 3 exists.



Proof 4



Proof 5



Proof 6

Proving

I have discussed the concept of proof, the texts that are called proofs, and now I will turn to the activity of proving. As the NCTM notes “a mathematical proof is a formal way of expressing particular kinds of reasoning” (2000, p. 56). But what kinds of reasoning should be called *proving*?

We have seen that in the twentieth century there was an emphasis in schools on a form of proofs that made their deductive structure very clear. In addition to the ways in which two column proofs helped teachers, their emphasis on deduction may also have derived from the focus on foundational questions in the first part of the century, which provided axiomatic structures for all branches of mathematics. The work of Polya and Lakatos later in the twentieth century critiqued this focus on axiomatics and deductive logic, pointing out the historical importance of analogies and empirical approaches in mathematical discovery and conjecturing. Their critiques, along with the growing awareness that teaching proving in schools was not very successful, led some jurisdictions, notably England, to put a greater emphasis on pattern noticing and conjecturing. The NCTM’s 1989 *Curriculum and Evaluation Standards* also placed conjecturing as an equal partner to verification by deductive reasoning. Hoyles (1997) points out that the emphasis on discovery and conjecturing in England meant that the majority of students saw nothing else. “The majority of students will engage in data generation, pattern recognition, and inductive methods while only a minority, at levels 7 or 8, are expected to prove their conjectures in any formal sense” (p. 9). Duval (1991) goes further, claiming that because arguments leading to conjectures and deductive proving “use very similar linguistic forms and propositional connectives,” an emphasis on discovery and conjecturing “is one of the main reasons why most of the students do not understand the requirements of mathematical proofs.” (p. 233). However, according to a group of Italian researchers, the relationship between conjecturing and proving is one of “cognitive unity” and their research has shown cases in which students built on what they had learned in making a conjecture to develop a deductive proof. Clearly, the relationship between conjecturing and proving deductively is unclear. However, it is

clear that there is a useful distinction to be made between these two kinds of reasoning. The particular kind of reasoning that is called proving is *deductive*. So it is especially important to clarify what deductive reasoning looks like.

When reasoning about a familiar context, even quite young students can produce deductive arguments. For example, consider this argument in favor of the existence of the tooth fairy made by five year olds:

Kim: My sister says there isn't even a tooth fairy. She says it's our mother.
Teacher: What do you think?
Kim: I think the fairy came in after my sister was asleep. Because my sister said I would get one dime and I got two dimes.
Kenny: I got a dollar. My mom can't spend a dollar because we are saving money for a car. So it has to be the tooth fairy.

(Paley, 1981, p. 40)

Kenny's deductions start with some prior knowledge: He knows he got a dollar for his lost tooth, and he knows that his mother is not spending any money because they are saving for a car. From this knowledge he deduces that his mother did not put the dollar under his pillow and so he rejects Kim's sister's conjecture in favor of Kim's, that the tooth fairy exists. Kenny does not include every step; he never says, "It can't be my mother". He also makes use of a hidden assumption that the only two possible sources of the tooth money are his mother and the tooth fairy. This is typical of children's deductions. Steps are skipped and hidden assumptions are made. But recall that mathematicians also skip steps and use shared assumptions without explicitly mentioning them. Kenny is far from producing a two column proof but he might not be so far from producing a proof acceptable to mathematicians.

Older children can link together simple deductions into chains in which conclusions of one step are used to justify subsequent steps. For example, consider Maya, age 11, who is explaining why the number of squares in a 10 by 10 grid is 10×10 + 9×9 + 8×8 + ... + 1 (Long dashes — represent pauses. Ellipses ... represent omissions).

Maya: Can everyone see? So you count 1, 2, 3, 4, 5, 6, 7, 8, 9 right? … Since a square—this—any square—the square is 10 by 10 no matter how you turn it, it's always going to be…the same. So you don't have to measure it again. You can go 9 times 9. Do you understand why? Yeah? OK, So you go 9 times 9 like Gino said, 81. Then you can do 3 by 3, 1, 2, 3, 4, 5, 6, 7, 8—and then again you don't have to measure again you know. It's going to be the same. So 8 times 8—64. And you can keep on going.… You can do the 7 times 7—49. And 6 times 6—36. 5 times 5—25. 4 times 4—16. 3 times 3—9.

(Reid & Knipping, 2010, p. 88)

First Maya counts how many 2 by 2 squares will fit along the top edge. Then she deduces that the number of 2 by 2 squares down either side will be the same, because the square is the same length on all sides. So the number of 2 by 2 squares is 9 times 9. For the 3 by 3 squares there are 8 along the top and the same reasoning gives 8 times 8. "And you can keep on going." The same reasoning gives the number of all the remaining sizes of squares.

Deductive reasoning is not something children need to be taught, although clearly contexts in which deductive reasoning is useful will encourage them to develop their reasoning further. However an important step toward proving is the formulation of that reasoning. It is necessary for students to become aware of their own reasoning and to explore ways to express that reasoning so that it can be understood by others. Maya is taking steps in that direction as she

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

explains her reasoning to her classmates. In the next section I will discuss ways to support he development of students' proving in classrooms.

I have discussed the role of proof, some kinds of proofs, and the nature of proving. I would like now to turn to the theme of the conference, transformative mathematics teaching and learning, and consider the teaching of proof. How might the goals of the NCTM's Reasoning and Proof Standard be realized? In what contexts do students learn to prove? How might teaching be transformed by a renewed attention to proof?

Teaching proof

Efforts to teach proof in the twentieth century occurred either in the context of a high school geometry course, or, in the New Math era, in a unit on logic. The NCTM is clear about one thing we learned from this experience:

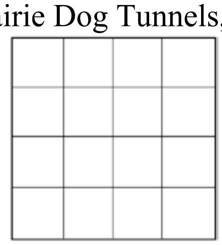
> Reasoning and proof cannot simply be taught in a single unit on logic, for example, or by "doing proofs" in geometry. (2000, p. 56)

But how *should* proof be taught? In this section I will consider first five aspects of teaching that Vicki Zack and I identified as contributing to students' proving in her classroom and in others'. I will then turn to two ideas that I believe help to clarify what it means to teach proof.

In Reid and Zack (2009) we describe how five key aspects of Vicki's teaching contributed to her grade 5 students' proving. They are: a focus on problem solving, allowing sufficient time, encouraging conjecturing, expectations concerning the nature of students' communications, and the teacher's expertise in listening closely and seizing opportunities to provoke discussion.

Problem Solving

Problem solving was at the core of the mathematics curriculum in Vicki's classroom. Non-routine problems were integrated into the everyday mathematics lessons, and a problem solving approach was also taken to other areas of the curriculum. The Problem of the Week tasks were especially rich as contexts for proving. These differ from the other non-routine problem-solving tasks used in the class in that the children were asked to write detailed explanations at two or more points in the problem solving process. For the most part Vicki used the same problems year after year. This allowed her to make small adjustments in phrasing or in follow up tasks that further supported the development of the children's reasoning (Brown, Reid & Zack, 1998). Four of the Problem of the Week tasks have received the most attention in our research. They are Prairie Dog Tunnels, Handshakes, Decagon Diagonals and Count the Squares (see Figure 1).



Find all the squares in the figure on the left. Can you prove that you have found them all?
What if ... ? [A five by five grid is given as a figure]. Can you prove you have them all?
Extension: What if it were a 10 by 10 square? What if it were a 60 by 60 square?

Figure 1: The Count the Squares problem

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

The prompts for Prairie Dog Tunnels, Handshakes, and Decagon Diagonals are:

- Nine prairie dogs need to connect all their burrows to one another in order to be sure they can evade their enemy, the ferret. How many tunnels do they need to build?
- If everyone in your class shakes hands with everyone else, how many handshakes would there be?
- How many diagonal lines can be drawn inside a figure with 10 sides? [Figures were provided of a triangle, square, pentagon and hexagon, labeled: 3 sides, 0 diagonals; 4 sides, 2 diagonals; 5 sides, 5 diagonals; 6 sides, 9 diagonals]
  How many diagonal lines would there be in a 25-sided polygon?
  How many diagonal lines ... in a 52 sided polygon?

In these tasks the students used common techniques such as making organized tables, using diagrams, and searching for patterns in sequences and using differences. The first three problems involve finding the sum for the whole numbers from 1 to n. In solving them the equivalence of $(1+2+3+4+\ldots+n–1)$ and $n(n–1)/2$ was discovered through a combination of empirical testing and deduction. Empirical strategies such as drawing all the possible diagonals were used to establish correct answers when n was small. These gave way (as n increased) to reasoning deductively either that each person shakes one less hand than the previous person (the first shakes $n–1$ hands, the next $n–2$, and so on) or that each of the n people shakes $n–1$ hands but that counts each handshake twice so the correct number is $n(n–1)/2$. The fact that these two methods of reasoning both correctly solve the problem established their equivalence.

Generally, the students did not spontaneously use symbols. Generalizations were most often expressed as procedures tied to the structure of the problem, for example, “Multiply the number of people by the number of handshakes each person does, which is one less, and divide by two because you counted all the handshakes twice.” Vicki then nudged the students to go further, asking them if they could express this idea by using a letter to represent any number of people shaking hands. In some years expressions like *n*(*n*–1)¸2 were written using variables. For example, Micky recorded this proof of a formula for finding the number of diagonals in a polygon:

> If you can find all the diagonals possible from one [vertex] you can figure out the whole amount of inside diagonal lines.... I know that a [vertex] connects with all of the other [vertices] except for 3, itself the [vertex] to the left and right. You subtract 3 from the amount of total sides, ... here’s the rule: (Z = no. of sides)...Z – 3 ´ Z ¸ 2 = no. of diagonal lines in figure.
>
> (from Micky’s Math Log, May 16, 1994, described in Zack, 1995)

Micky’s proof has several steps. He is explicit about one: that the number of diagonals from each vertex is Z – 3 (because each vertex is connected by diagonals to all the others, except itself and its neighbors). However, he leaves the justification for another step implicit: Every diagonal is counted twice, so the formula ends with dividing by two.

Time

In Vicki’s class the children were given a great deal of time to experiment with, think through, discuss and refine their understandings. Each Problem of the Week was assigned on a Monday. The students worked independently on the problem during class time (an extended 90 minute class period) and wrote a detailed description in their Math Logs. On Tuesday, she allowed time for the students to review their Logs if they chose to do so and at times prompted

some students to reflect upon their explanation (for example, to clarify their thinking, elaborate on a diagram or add a diagram). Wednesday was the day the discussions took place, again in a 90-minute session. They worked first in pairs (or in a group of three), and then came together in a group of four or five. In these small groups they compared solutions and discussed further, and then reported to the half-class, with more discussion following. On occasion, if the discussions in the small group or the class warranted, class time on Thursday and Friday was also used to allow the discussions to come to a fruitful conclusion. This means that the exploration of the problems occurred over a significant period of time, almost four hours per task. This allowed for conjectures to be made and explanations sought without being artificially cut short by time constraints.

Conjecturing

The problems used allowed the students to make hypotheses and discover solutions, which they then proved in order to verify and explain. The processes of conjecturing and proving were intertwined in two ways. Proving made use of insights gained through the explorations that led to conjectures (For example, cut-out squares of different sizes were used to support the counting of squares, but they were also used in proving. See Maya's proving above and Zack, 2002). Conjectures were also used as the basis for proving (For example, one child conjectured that the number of squares would always be a multiple of 5, unless the size of the grid was a multiple of 3. He later rejected answers conjectured by his peers if they contradicted his general rule; see Reid, 2002).

Expectations

The groundwork for proving laid during the year included an expectation that the children would be looking for patterns, and that they could be nudged to think about the mathematical structure underlying the pattern (Zack, 1997). In addition, there were expectations that everyone's answers should be considered and that answers should not be changed without discussing how they arose and what might be the source of an error.

This valuing of clarity, explanation, and attention to others supported the expression of the students' thinking, whether it involved proving or not, but also made that thinking available to others to question. At the close of each session, Vicki distributed a sheet entitled "Helpful explanations/Helpful ideas" and asked the children to note any ideas or explanations they found helpful, to tell why, and to credit the peer(s) who helped. She found that the children became increasingly aware of the contribution others had made to their understanding, and at times could indicate how they have reshaped others' ideas to make them their own (See Zack & Graves, 2001 for further elaboration).

Expertise

In terms of her mathematical background, Vicki could be considered a typical elementary school teacher in that she describes her background in formal mathematics as weak. She is not an expert on the nature and role of mathematical proof, and has not formally proven anything herself since high school. However, her research interest in how meaning is constructed in dialogue led to a close look at the children's ways of expressing their ideas, and then in turn at issues of convincing and proving.

Working with the children's ways of making meaning is central to Vicki's teaching. Her expertise lay in listening closely, recognizing potentially fruitful avenues and seizing opportunities to provoke discussion. There was a constant expectation for explanation (e.g.,

asking "Explain how it works and why it works."), and for generalization (Will it always work? Can you construct a general rule?). Vicki discovered the richness of the mathematical tasks in large measure due to the children's ways of solving the problem. The tasks lent themselves to algebraic thinking and generalizing and proving. The proving arose from the tasks, even though there had been no 'a priori' objective to teach proof and proving.

The five aspects of teaching described above, problem solving, sufficient time, conjecturing, expectations for communication, and teacher expertise, seem to be important in creating a context for proving, even where no explicit intent to teach proof existed. However, in higher grades where the teaching of proof might become more structured, more attention must be paid to two key ideas: the tool-box, and local organization.

The Tool-box

Mathematicians' and children's proofs make reference to, or assume without stating, other theorems or assumptions. Things are taken for granted. Even in the work of professional mathematicians there are theorems that are used without their proofs being read, and even without any source being known.

> Within any field, there are certain theorems and certain techniques that are generally known and generally accepted. When you write a paper, you refer to these without proof. You look at other papers in the field, and you see what facts they quote without proof, and what they cite in their bibliography. You learn from other people some idea of the proofs. Then you're free to quote the same theorem and cite the same citations. You don't necessarily have to read the full papers or books that are in your bibliography. Many of the things that are generally known are things for which there may be no known written source. As long as people in the field are comfortable that the idea works, it doesn't need to have a formal written source. (Thurston, 1995, p. 33)

Netz (1999) discusses the omission of references to theorems and assumptions in classical Greek proofs. He calls the set of theorems and assumptions that can be used without comment the "tool-box". For example, Archimedes can assert that two segments are of the same length because they are radii of the same circle. He does not have to make any reference to this justification. He simply states that they are the same length and leaves it to the reader to figure out why (Netz, 1999, p. 172). As his readers were all members of a cultural community for whom the same tool-box was taken for granted, this was acceptable.

Similarly, Proofs 1, 2, 4 and 5, above, make use of tools from my tool-box which includes rules governing algebraic symbolization and manipulations, the idea of mathematical induction, and even implicit rules about using generic examples that have probably never been stated anywhere, but which are followed by those who consider such proofs acceptable.

Netz argues that the results proven in Euclid's Elements constitute most of the classical Greek tool-box. One might imagine that the study of the Elements was undertaken as the beginning of a scholar's mathematical education and so thereafter mathematicians could assume that anything in Euclid could be used without comment or reference. However, Netz suggests that the contents of the tool-box could become known in another way.

> The very fact that an argument was made, without any intuitive or diagrammatic support for that argument, must have signalled for the audience that the argument was sanctioned by the Elements. Once this is the expectation, the need to refer explicitly to the Elements declines, which would in turn support the same tendency: the regular circle in which local conventions are struck without explicit codification. (p. 232)

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

In other words, the process might resemble that described by Thurston, above; the reader comes to know, by observing what things are taken for granted in proofs, what is in the tool-box, without even being told explicitly.

In a mathematics classroom, determining what is in the class tool-box can be done in various ways. One approach sometimes taken is to inform the students that they must forget everything they already know and start from the axioms and definitions given in the textbook. But experience tells us that the students do not forget everything they know, and the axioms and definitions given in the textbook are never complete themselves. What results is the class pretending to base its arguments on the given axioms and definitions, while being guided by their prior knowledge. The tool-box is supposed to be limited to what the textbook allows, but is actually larger.

Another approach is for a teacher to start presenting proofs without establishing what is in the tool-box, so that, as Netz says happened for the Greeks, the contents of the tool-box become known to the students through the making of arguments without stating their justifications. If a proof is based on an assumption that the measures of angles can be added, but this is never stated or justified, then it must be part of the tool-box.

The idea of the tool-box can be usefully connected with another idea, that of local organization.

Local Organization

Freudenthal (1971) discusses the introduction of proving and claims that proving must begin with what he calls "local organization" as opposed to the "global organization" of an axiomatic system. In a globally organized system the definition of parallelogram would be part of the tool-box and would either be explicitly taught or would become known through its use in proofs. Freudenthal describes another approach: A discussion, for example, of the properties of parallelograms can begin by simply listing all those that are apparent to the students. Similar lists might be made for rectangles and rhombuses. In examining such lists, Freudenthal claims, "There are a host of visual properties which ask for organization. Here starts deductivity; rather than being imposed it unfolds from local germs. The properties of the parallelogramme become deductively interrelated" (p. 424). Finally, one property emerges as a definition from which the others can be deduced. This is local organization. It can be extended as the properties of parallelograms are related to the properties of rectangles, rhombuses and squares.

So rather than a definition being given at the outset, the students' proving determines which property is a definition and which ones are consequences of it. There is a local organization of mathematical knowledge, but no global system into which that knowledge fits. This is necessary, Freudenthal states, because, "a student who never exercized organizing a subject matter on local levels will not succeed on the global one" (p. 426). It is at the local level that proving and defining are learned, before being used (perhaps much later) to define and prove in an axiomatic system.

Freudenthal's implications for teaching are clear:

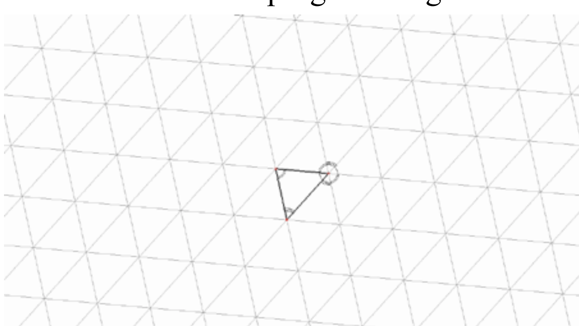
> In general, what we do if we create and if we apply mathematics, is an activity of local organization. Beginners in mathematics cannot do even more than that. Every teacher knows that most students can produce and understand only short deduction chains. They cannot grasp long proofs as a whole, and still less can they view substantial part of mathematics as a deductive system. (p. 431)

Local organization and use of an implicit tool-box raise the question for teaching of what must be made explicit, what requires proof, and what can be left implicit. Freudenthal's

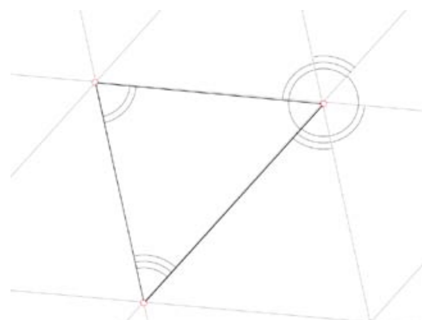
examples (in *Mathematics as an Educational Task*, 1973) offer a resolution to this problem. The decision as to what to leave implicit, what to make explicit and what to prove must be made in order to develop students' understanding of mathematical concepts and ability to apply them. In cases where the proof brings a new understanding of the concepts involved, the proof is useful. Assumptions that have unexpected implications will need to be made explicit in exploring those implications. Assumptions that would seem obvious and trivial to the students if made explicit can safely be left implicit.

Proof 7 is another visual proof that the angle sum of a triangle is 180 degrees. It depends on an assumption that any triangle can be used to tile the plane. It is clear that the six angles around a vertex total 360 degrees (although this depends an another hidden assumption, that angle measures can be added, and a hidden definition, of degrees). The congruence of the marked angles depends on the fact that all the triangles are congruent. All of the hidden assumptions here are likely to be accepted without question by students, and making them explicit would be pointing out the obvious. In contrast, the typical two column proof depends on properties of transversals that are not so obvious as assumptions, and if proven depend on the even less obvious parallel postulate.

Proof 7 makes use of a minimal tool-box, and puts together its elements in a way that proves an important mathematical property of plane triangles, a property which in turn contributes to many other proofs of significant results. As it is based on assumptions that are more obvious than the conclusion it verifies and explains the result, in a way that the traditional two column proof does not. A similar comment could be made about Proof 4, although unless it is actually enacted the fact that one turns 360 degrees in tracing around a triangle may not be obvious to those who have not done LOGO programming.



Proof 7



Proof 7 detail

Teaching Proof versus Proof Based Teaching

The NCTM opened the twenty-first century with a call for increased attention on mathematical proof and reasoning, for all students. Reform movements at the beginning of the twentieth century also called for mathematics teaching, and especially high school geometry teaching, to focus much more on proof, at a time when parallel reforms were allowing many more students to go to high school. In North America this resulted in the invention of a new proof format, the two column proof, along with other pedagogical innovations designed to help teachers teach proof.

> To make student proving possible, a system of resources had to be developed and coordinated with a norm for accomplished proofs. The integration of all those elements produced a stable geometry course oriented toward students' learning the art of proving embodied in the two-column format. However, that stability came with a price – that of

> dissociating the doing of proofs from the construction of knowledge. (Herbst, 2002, p. 307)

Instead of school proofs being like mathematicians proofs in that they gave insight into mathematics, school proofs became another topic to be covered, disconnected from all others. Proofs became an object, rather than a process of learning.

To avoid following the same path now, we must tread carefully. We must ensure that we see proof as fundamental to mathematics as a way to develop understanding of mathematical concepts, and as a way to discover new and significant mathematical knowledge. Proof cannot be limited to the format of proofs, and to the role of verification of knowledge (for which there is probably good empirical or other evidence already).

We have recent experience with a comparable reform effort, the increased focus on mathematical problem solving after the 1970s. At first, problem solving was an object of teaching. Texts added new chapters on problem solving, including Polya's four stages, sets of heuristics and even larger sets of strategies to help teachers and students learn to be better problem solvers. Problem solving was a huge area for research in mathematics education. Unfortunately, a lot of the research indicated that making problem solving an object of mathematics teaching wasn't making anyone better problem solvers (Schoenfeld, 1987, p. 30). Perhaps the people with the best success in developing problem solving in students, the mathematics faculty who coached students for the Putnam exam or for various Olympiads, had it right. "Students don't learn to solve problems by reading Polya's books, they said. In their experience, students learned to solve problems by ... solving lots of problems." (p. 30). Some decades on now, problem solving is taking on a different role in mathematics teaching. Textbooks for new teachers and some curricula and school mathematics textbooks are advocating that problems not be an object of mathematics teaching, but instead be the means by which mathematics is taught.

I would like to conclude by suggesting that today proof teaching and research is in a similar state to problem solving in the 1980s. Much research is being done, but proof is still seen as a topic to be taught. In the case of problem solving we have moved towards seeing it as a way of teaching, and there are early hints that something like proof based teaching, akin to problem based teaching, might emerge in the next decades. It will be interesting to see.

References

Brown, L., Reid, D., & Zack, V. (1998). On doing the 'same problem.' *Mathematics Teaching, 163*, 50-55.

Davis, P. (1972). Fidelity in mathematical discourse: Is one and one really two? *American Mathematical Monthly, 79*, 252-263.

Davis, P. J., & Hersh, R. (1981). The mathematical experience. Boston: Birkhäuser.

Descartes, R. (1993). Discourse on method and meditations on first philosophy. (D. Cress, Trans.). Indianapolis: Hackett Publishing. (Original work published 1637).

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration. *Educational Studies in Mathematics, 22*(3), 233-261.

Fischbein, E., & Kedem, I. (1982). Proof and certitude in the development of mathematical thinking. In A. Vermandel (Ed.), Proceedings of the Sixth Annual Conference of the International Group for the Psychology of Mathematics Education, (pp. 128-131). Antwerp.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics, 3* (3-4), 413-435.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.
Hanna, G. (1983). Rigorous proof in mathematics education. Toronto: OISE Press.
Hanna, G. (1989). Proofs that prove and proofs that explain. In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), Proceedings of the Thirteenth International Conference on the Psychology of Mathematics Education, (Vol. 2, pp. 45-51). Paris: Program Committee.
Herbst, P. (2002). Establishing a custom of proving in American school geometry: Evolution of the two-column proof in the early twentieth century. *Educational Studies in Mathematics, 49,* 283-312.
Hoyles, C. (1997). The curricular shaping of students' approaches to proof. *For the Learning of Mathematics 17*(1), 7-16.
Joseph, G. G. (1991). *The crest of the peacock: Non-European roots of mathematics.* London: Penguin.
Krantz, S. (1989). Fractal geometry. *Mathematical Intelligencer, 11*(4), pp. 12-16.
Maxwell, E. A. (1959). *Fallacies in mathematics.* Cambridge UK: Cambridge University Press.
National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.
Netz, R. (1999). The shaping of deduction in Greek mathematics: A study in cognitive history. Cambridge UK: Cambridge University Press.
Paley, V. (1981). *Wally's Stories.* Cambridge, MA : Harvard University Press.
Reid, D. & Zack, V. (2009) Aspects of teaching proving in upper elementary school. In D. Stylianou, M. Blanton & E. Knuth (Eds.) *Teaching and learning proof across the grades*. New York: Routledge.
Reid, D. A. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education, 33*(1), 5-29.
Reid, D., & Knipping, C. (2010). *Proof in mathematics education: Research, learning and teaching.* Rotterdam: Sense.
Schoenfeld, A. (1987). Confessions of an accidental theorist. *For the Learning of Mathematics, 7*(1), 30-38.
Thurston, W. (1995). On proof and progress in mathematics. *For the Learning of Mathematics, 15*(1), 29-37. Originally published in *Bulletin of the American Mathematical Society, n.s., 30*(2), 161-177.
Zack, V. (1995). Algebraic thinking in the upper elementary school: The role of collaboration in making meaning of 'generalization'. In L. Meira and D. Carraher, (Eds.), *Proceedings of the Nineteenth Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 106-113). Recife, Brazil: Program Committee.
Zack, V. (1997). "You have to prove us wrong": Proof at the elementary school level. In E. Pehkonen (Ed.), *Proceedings of the Twenty-First Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 291-298). Lahti, Finland: Program Committee.
Zack, V. (2002). Learning from learners: Robust counterarguments in fifth graders' talk about reasoning and proving. In A. D. Cockburn and E. Nardi (Eds.), *Proceedings of the Twenty-Sixth International Conference for the Psychology of Mathematics Education* (Vol. 4, pp. 433-441), Norwich, UK: Program Committee.

Zack, V. and Graves, B. (2001). Making mathematical meaning through dialogue: "Once you think of it, the z minus three seems pretty weird." In C. Kieran, E. Forman & A. Sfard (Eds.), *Bridging the individual and the social: Discursive approaches to research in mathematics education*. Special Issue, *Educational Studies in Mathematic*s, *46*(1-3), 229-271.

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.